ISSN 2318-2377

Cedeplar
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TEXTO PARA DISCUSSÃO Nº 665

LARGE PUBLIC EXPENDITURE SHOCKS IN A RAMSEY TAXATION MODEL WITH DEFAULT

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Dezembro 2023

http://www.cedeplar.ufmg.br

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Ficha Catalográfica

G184l
2023

Larde public expenditure shocks in a Ramsey taxation model with default / Juan Pablo Gama, Rodrigo Jardim Raad. - Belo Horizonte: UFMG / CEDEPLAR, 2023.

p. - (Texto para discussão, 665)

Inclui bibliografia.

1. Inadimplência (Finanças). 2. Brasil - Condições econômicas.

I. Juan P. Gama. II. Rodrigo J. Raad. III. Universidade Federal de Minas Gerais. Centro de Desenvolvimento e Planejamento Regional. IV. Título. IV. Série.

CDD: 332

Elaborado por Adriana Kelly Rodrigues CRB-6/2572
Biblioteca da FACE/UFMG. – AKR/165/2023

UNIVERSIDADE FEDERAL DE MINAS GERAIS
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BELO HORIZONTE
2023

SUMÁRIO

RESUMO

Este artigo analisa um modelo de Ramsey de três períodos com agentes e um título oferecido por um planejador central. Nós mostramos que a propriedade fiscal de Ramsey mantém, ou seja, as taxas de imposto são mais altas para mercados com menor elasticidade-preço na alocação social ótima. Além disso, mostramos que uma estratégia ótima do planejador central baseada apenas em caso de inadimplência parcial no pagamento de juros implementa a alocação de bens com tributação. Portanto, um aumento na despesa pública devido a uma crise de saúde pública como a COVID-19 não pode ser financiado por um calote parcial da dívida pública. Na verdade, sobrecarrega os agentes que consomem maior quantidade de bens inelásticos, ou seja, aqueles com baixos rendimentos. Concluímos que uma despesa de emergência deve ser financiada através de um aumento dos impostos sobre os bens finais, poupanças preventivas ou impostos sobre o rendimento[1] que não participam directamente no planeamento de emergência de um sistema de saúde pública em crise.

Tributação ótima, Equilíbrio geral, Inadimplência, Choques nas despesas públicas

ABSTRACT

This paper analyses a three-period Ramsey's model with heterogeneous agents and a bond offered by a central planner. We show that Ramsey's taxation property holds, that is, tax rates are higher for markets with lower price elasticity in the social optimal allocation. Additionally, we show that a central planner optimal strategy based only on a partial default on interest payments implements the former goods allocation with taxation. Therefore, an increment on public expenditure due to a public health crisis such as the COVID-19 cannot be financed by a partial default on the public debt. Indeed, it overburdens agents who consume larger amount of inelastic goods, that is, those with lower income. We conclude that an emergency expenditure must be financed through an increase of Value-Added taxes, precautionary savings, or income taxes[2] who do not participate directly on the emergency planning of a public health crisis.

Keywords Optimal taxation, General equilibrium, Default, Public expenditure shocks, COVID-19.

Código JEL: D50, D52

[1] Ou uma redução em despesas primárias semelhantes do governo.
[2] Or a reduction on similar government primary expenditures.

1 Introduction

The first action has a strong economic and social impact due to voluntary or compulsory isolation or complete lock-down in the most extreme cases. The latter actions have strong consequences on the reduction of the mortality rate due to complications or other diseases that the health system is unable to treat properly due to its collapse. However, all these actions have a strong fiscal impact which causes a fast increase on government expenditures and a reduction on tax revenue in the short-run (Ozili and Arun, 2020).

Most countries finance these emergency expenditures with public bonds or international loans therefore increasing the fiscal instability in the future (Reinhart et al., 2015). Additionally, some of these countries must finance themselves with debt at high interest rates increasing their chances of default in the future (Ozili and Arun, 2020). This frequently occurs in developing countries (Loayza and Pennings, 2020) such as Argentina and Brazil which are tempted to default at least partially.

Optimal taxation models generally do not address global heterogeneity (Chamley, 1986) because the inclusion of a technology that transforms consumer goods into capital as a constant proportion is equivalent to consider capital price as exogenous[3]. However, the endogeneity of the asset prices and the consumers' preferences might play a key role on how the central planner's default contaminates prices. Therefore, these results in the literature cannot be used to explain how prices will be contaminated by central planners' default. Additionally, despite some authors such as Auerbach (1979) introduce heterogeneous capital, they do not include heterogeneity on preferences.

Ramsey (1927) based models have been wide applied to optimal taxation theory including the seminal works of Diamond and Mirrlees (1971) and Aiyagari (1995). More recently, Acikgöz et al. (2018) and Chien and Lee (2016) analyze optimal capital taxation in infinite horizon models with incomplete markets and endogenous government spending respectively. In both cases, the central planner imposes capital and labor income taxes in the presence of heterogeneous consumers in economies with a countable number of dates. However, the central planner is not able to anticipate the aggregate demand function in the Subgame Perfect Nash Equilibrium to estimate its optimal taxes.

In this paper, we consider public expenditures as endogenous but bounded away from zero due to an extreme negative shock which leads to a market failure covered only by a centralized social security. Even when there is only future public expenditures and no current ones, the central planner can coexist with the private sector due to informational asymmetries. The inclusion of an extreme negative shock implies the existence of a persistent market failure where centralized decisions can implement a Pareto improvement. Rationality leads market agents to anticipate the extreme negative shock by changing current decisions, which justifies the Pareto improvement in centralized contracts even for long run extreme shocks.

We consider a sequential game with a Walrasian auctioneer, a central planner, and consumers. The latter choose a best response demand with price and a fiscal policy taken as given. The central planner can estimate the aggregate demand function in the Subgame Perfect Nash Equilibrium.[4] However, it does not know specifically which equilibrium price will occur due to multiplicity of equilibria. Therefore, the central planner chooses its optimal fiscal policy as a best response function based on the aggregate

[3] In our model, the absence of production makes that private capital investment almost identical to public bonds from the consumers' perspectives.

[4] The central planner could use historic data to anticipate demand.

demand, but contingent on equilibrium prices. The Walrasian auctioneer takes all best responses as given and chooses prices of equilibrium as a best response. Therefore, optimal choices in this game could be viewed as a Subgame Perfect Nash Equilibrium.

We show that, in a three-period Ramsey's model with heterogeneous agents, no production, and a bond offered by the central planner, Ramsey's taxation property holds, that is, taxes are higher for markets with less price elasticity of the aggregate demand. Therefore, increments on the central planner expenditures will impact more the inelastic markets than the elastic ones. As a consequence, agents who spend a larger proportion of their wealth in the inelastic goods, mostly poor agents, will be the ones who receive a higher tax burden.

We also show that a default of part of the central planner bond has no real effect on the demand. This result is a consequence of the consumers' rationality which induces markets to anticipate of public default risk. By doing so, the market chooses a price that reflects the default, and agents adjust their consumption based on this information. This is particularly relevant when the social planner is forced to increase radically its expenditure to finance emergency costs as in the current health crisis caused by the pandemic of COVID-19. As a consequence, a central planner optimal strategy must avoid default as much as possible due to the lack of benefits that it will have. Moreover, it will create two negative signals to the market: the default itself of part of its debt, and a reduction of the public bond price in the short-run. The former is straightforward since markets anticipate a partial default and hence equilibrium interest rates increase. The latter is a clear signal for the investors that the central planner bond is less attractive increasing the probability of a currency crisis in the short-run. At the same time, if the economy uses the public bond to index all prices, there will be a large increment of the price of real goods emulating an "inflationary" effect due to a high debt and its future default.

We finally show that in the case of income taxes in a model with public employees, optimal taxation is higher on public employees whose wages do not compensate the social benefit of their services or produced goods. This is possibly the case of public employees that have wages higher than those which are offered in the private sector or employees that has little or no effect on how to overcome the public health crisis analyzed.

Therefore, the best strategy a central planner could choose is to finance this mandatory extra expenditure with value-added taxes in long-run if an increment of income taxes are not available. Unfortunately, this taxes will load the poorest agents strongly due to their larger demand of inelastic goods. If an increment of income taxes is available, high-paid public employees will finance most of the extra expenditures that are required.

The possibility of having a default in the second period (short-run) is not particularly relevant for this paper since we currently study the impact on demand and welfare when the central planner faces a public health crisis such as the COVID-19. In these cases, countries increase dramatically their public debt to reduce the negative social and economic impact even in countries with high levels of public debt such as Italy, Portugal, United States, and Brazil, or countries with high chances to default such as Argentina[5]. Moreover, we are more interested in the long-run consequences of the increment of public debt and the central planner strategy to pay it back.

[5] In this case, they negotiate their debt with creditors to pay it back in long-run with a predefined default (see Alzúa and Gosis (2020)).

The article is organized as follows. In section 2, we give some basic notation. In section 3, we define the model that we consider. In section 4, we show the main results. Finally, in section 5, we give some concluding remarks.

2 Basic Notation

We will start with some basic notation conventions first. Define

$$T = \{0, 1, 2, \cdots, \mathrm{t}\} \text{ and } K = \{0, 1, 2, \cdots, \mathrm{k}\}.$$

In this paper, we work on the following vector spaces, say, $\mathbb{R}$, $\mathbb{R}^{\mathrm{t}}$, $\mathbb{R}^{\mathrm{k}}$,

$$\mathbb{R}^{\mathrm{t}\times\mathrm{k}} = \{z_{tk} : (t, k) \in T \times K\} \qquad \mathbb{R}^{\mathrm{k}\times\mathrm{t}} = \{z_{kt} : (k, t) \in K \times T\}$$

and

$$\mathbb{R}^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}} = \{z_{tk\kappa} : (t, k, \kappa) \in T \times K \times K\}.$$

We call a linear space L any element of these spaces. An element $z \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ is a matrix with t rows and k columns. Given matrices $x \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ and $y \in \mathbb{R}^{\mathrm{k}\times\mathrm{n}}$ we write $xy \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ for the standard matrix product. Given $z \in L$ and $\dot{z} \in L$, write $\mathring{z} = \dot{z} - z$ as the differential direction or increment.

Let us write any price or tax vector at period $t \in T$ as a row matrix $z_t = [z_{t1}, \cdots, z_{t\mathrm{k}}]$. Denote by $z \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ the matrix of price or tax streams. Write the vector of allocations at period $t \in T$ as a column matrix[6] $z_t = (p_{1t}, \cdots, p_{\mathrm{k}t})$. Denote by $z \in \mathbb{R}^{\mathrm{k}\times\mathrm{t}}$ the matrix of allocation streams.

Given two linear spaces L and L' we define the derivative of a function $f : A \subset L \to L'$ where A is open at a point z as a linear function denoted by $Df(z) : L \to L'$ defined in the standard[7] way.

Given the linear spaces (L, L', L'') and $f : L \times L' \to L''$, we denote the partial derivative with respect to $z \in Z$ as

$$\partial_1 f(z, z')(\mathring{z}) = Df(z, z')(\mathring{z}, 0) \text{ for all } \mathring{z} \in L.$$

Consider $f : L \times L' \to L''$ a bilinear function. Then[8]

$$Df(z, z')(\mathring{z}, \mathring{z}') = f(z, \mathring{z}') + f(\mathring{z}, z') \tag{1}$$

The Chain Rule is denoted as follows. Consider $f : Z \subset L \to L'$ and $g : L' \to L''$. Set $h : Z \subset L \to L''$ by $h(z) = g(f(z))$ for all $z \in Z$. Then given $z \in Z$ then

$$Dh(z)(\mathring{z}) = Dg(f(z))(Df(z)(\mathring{z})) \text{ for all } \mathring{z} \in L. \tag{2}$$

We write the letter with the symbol "$\widehat{\ }$" to distinguish a function from its variable. For example $x = \hat{x}(p, w)$ means the value x of the function $\hat{x}$ evaluated at the point (p, w). For notation purposes, let us denote the symbol without upper index as the Cartesian product, for example, for a set of J agents, write

[6] We denote by $(\cdots)$ the column matrix and by $[\cdots]$ the row matrix.
[7] That is, the Fréchet derivative.
[8] See Bartle (Bartle) for more details.

$Z = \prod_{j\in J} Z^j$ and analogously for elements and functions. Moreover, denote the symbol "$+$" in the upper index to stand for aggregate variables over markets excluding the central planner. For example denote $z^+ = \sum_{i\in I} z^i$, the aggregate variable on a set of I agents. Finally, denote generically by $\mathring{z}(t,k) \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ as a direction which is zero in all but the tk-coordinate. Write $\mathring{z}(t,k) \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ as a unitary increment when $\mathring{z}_{s\kappa}(t,k) = 0$ for $(s,\kappa) \neq (t,k)$ and $\mathring{z}^i_{tk}(t,k) = 1$. Therefore, given $z \in Z$ $\dot{z} \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ and $f : Z \subset \mathbb{R}^{\mathrm{t}\times\mathrm{k}} \to L$

$$Df(z)(\dot{z}) = \sum_{(t,k)\in T\times K} \dot{z}_{tk} Df(z)(\mathring{z}(t,k))$$

because $\dot{z} = \sum_{(t,k)\in T\times K} \dot{z}_{tk}\mathring{z}(t,k)$.

3 Model

Let us define a Ramsey type of economy with a finite number of agents indexed in the set I and a central planner indexed by c. Let $J = I \cup \{c\}$ the set of all agents. Suppose an environment with three periods without uncertainty[9] indexed in the set $T := \{0,1,2\}$, three goods in each period indexed in the set $K := \{0,1,2\}$ where the good $k = 0$ is a *numéraire* good, and a bond traded by the social planner with amount denoted by $a^c \in \mathbb{R}^3_+$ and real interest given by[10] $r \in \mathbb{R}^{\mathrm{k}\times\mathrm{t}}_{++}$ with $r = [r_t]_{t\in T}$ and r_t is the column matrix $r_t = (r_{kt})_{k\in K} \in \mathbb{R}^{\mathrm{k}}_+$ for each $t \in T$. Write the row matrix $p_t = [p_{tk}]_{k\in K} \in \mathbb{R}^3_+$ as the good prices and $q_t \in \mathbb{R}_+$ as the asset prices for each period $t \in T$. Define the matrix $p = (p_t)_{t\in T} \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}$ as the stream of good prices and $q = [q_t]_{t\in T}$ analogously. Suppose that *numéraire* good has unitary price, that is, $p_{0t} = 1$ for each $t \in T$. Write the set of good prices as

$$P = \{[p_t]_{t\in T} \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}} \text{ such that } p_{0t} = 1 \text{ for each } t \in T\}$$

and the set of asset prices as $Q = \mathbb{R}^3_+$. Regarding tax notation, write the matrix of taxes at each period $\tau_t \in \mathbb{R}^{\mathrm{k}\times\mathrm{k}}$ with a typical element $\tau_{t\kappa\ell} = 0$ if $\kappa \neq \ell$ and $\tau_{tkk} \in \mathbb{R}_+$ representing the tax of good k at time t. Denote by $\tau = (\tau_t)_{t\in T}$ the array of taxes and τ_{tkk} by τ_{tk} to simplify. An agent i's allocation is written as a matrix $x^i \in X^i := \mathbb{R}^{\mathrm{k}\times\mathrm{t}}$ in which x^i_{kt} represents the consumption of good $k \in K$ at period $t \in T$. Write the consumption column vector as $x^i_t = (x^i_{kt})_{k\in K}$ for each $i \in I$. Therefore, the amount $p_t\tau_t x^i_t \in \mathbb{R}_+$ represents the total tax Agent i pays for consuming x^i_t. Consumers choose the amount $a^i \in \mathbb{R}^{\mathrm{t}}$ of bonds and the set of consumer choices is then given by

$$Y^i = \mathbb{R}^{\mathrm{k}\times\mathrm{t}}_+ \times \mathbb{R}^{\mathrm{t}}_+$$

with a typical element $y^i = (x^i, a^i)$. In this model, the central planner has enough information about the economy to know the consumers' demand, but it does not have sufficient information to know which equilibria will occur. Therefore, it behaves as price taker similarly to consumers but it defines market bonds supply,[11] taxes and social expenses.

[9] All results id this model hold in framework including an exogenous uncertainty but we omit this structure for the sake of simplicity.

[10] Note that fiat money is not traded in this model since it is assumed that there are no transaction costs and there is no taxation paid with it (and hence agents have no incentive to use it for tax evasion purposes).

[11] The central planner control bonds supply by setting suitable interest rates.

The central planner set of choice or fiscal policy is given by

$$Y^c = \mathbb{R}_+^{\mathrm{k}\times\mathrm{t}} \times \mathbb{R}_+^{\mathrm{t}} \times \mathbb{R}_+^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}} \times \mathbb{R}_+^{\mathrm{k}\times\mathrm{t}}$$

where any fiscal policy $y^c = (x^c, a^c, \tau, r)$ is given by a marginal taxation τ_t in period $t \in T$, an asset supply a_t^c at date $t \in T$, an interest rate r_t, and public expenses x^c relying on emergency demand. We suppose that $\tau_{0t} = 0$ for all $t \in T$. As it will be explained briefly, taxes will be imposed on prices of real goods in the form of Value-Added Taxes (VAT). Additionally, we suppose that $p_{0t} = 1$ for all $t \in T$ without loss of generality.

3.1 Consumer problem

The consumption set of an agent i is given by $Y^i = \mathbb{R}_+^{\mathrm{k}\times\mathrm{t}} \times \mathbb{R}_+^{\mathrm{t}}$. A consumption plan $y = (x^i, a^i)$, also written as $y = (x_0^i, x_1^i, x_2^i, a_0^i, a_1^i, a_2^i)$, is given by consumption goods, $x_t^i \in \mathbb{R}_+^{\mathrm{k}}$, and an asset purchase, $a_{t-1}^i \in \mathbb{R}_+$, at period $t \in T$ with a_{-1}^i satisfying $\sum_{i\in I} a_{-1}^i$ as the *ex-ante* total of public debt. Denote by $e^i \in \mathbb{R}^{\mathrm{k}\times\mathrm{t}}$ the good endowment stream for each $i \in I$.

Assume that taxes are levied on endowments and net purchase but not on net selling. Therefore, if an agent i has endowment e_t^i at period t and consume x_t^i then the net purchase is $n_t^i = x_t^i - e_t^i$. If $n_t^i \geq 0$ then agent i pays $\tau_t n_t$ from net purchase plus $\tau_t e_t^i$ from endowments. Thus the total tax amount is given by $\tau_t(n_t + e_t^i) = \tau_t x_t^i$. If $n_t^i < 0$ then agent i sells $|n_t^i|$ and keeps x_t^i as endowment.[12] Therefore in both cases the total tax amount is given by $\tau_t x_t^i$. Given an interest rate d, the budget constraint of an agent $i \in I$ is given by

$$\hat{b}^i(p, q, \tau, r, e) = \left\{ y^i \in Y^i : p_t x_t^i + q_t a_t^i \leq p_t r_t a_{t-1}^i + p_t e_t^i - p_t \tau_t x_t^i \; \forall t \in T \right\}$$

where $e_t^i \in \mathbb{R}_+^3$ is the initial allocation of the agent i at date t. The utility function $u^i : \mathbb{R}^{\mathrm{k}\times\mathrm{t}} \to \mathbb{R}$ is given by $u^i(x^i) := \sum_{t\in T} \beta^t u_t^i(x_t^i)$ where $u_t^i : \mathbb{R}_+^3 \to \mathbb{R}$ is the utility index for $t \in T$. We also assume that u_t^i is a strictly increasing and concave C^1 function such that[13]

$$u_t^i(x_t^i) = \sum_{k\in K} u_{kt}^i(x_{kt}^i) \text{ for all } x_{kt} \in \mathbb{R}_+.$$

The indirect utility function is given by

$$v^i(p, q, \tau, r, e) = \max \left\{ \sum_{t\in T} \beta^t u_t^i(x_t^i) : (x^i, a^i) \in \hat{b}^i(p, q, \tau, r, e) \right\} \tag{3}$$

and the solution of the consumer problem given by

$$\hat{y}^i(p, q, \tau, r, e) = \operatorname{argmax} \left\{ \sum_{t\in T} \beta^t u_t^i(x_t^i) : (x^i, a^i) \in \hat{b}^i(p, q, \tau, r, e) \right\}.$$

[12] There is no net sale taxation.

[13] We can consider exogenous uncertainty in this model and define u_t^i as the V.N.M. expected utility. But we do not model exogenous uncertainty for the sake of simplicity.

Write it shortly as $\hat{y}^i(p,q,\tau,r,e) = (\hat{x}^i(p,q,\tau,r,e), \hat{a}^i(p,q,\tau,r,e))$ for each $y^c \in Y^c$. Therefore, $\hat{y}^i$ is the agent i's optimal strategy.

3.2 Central planner problem

Write a vector of fiscal policy as

$$y^c = (x^c, a^c, \tau, r) = (x_0^c, x_1^c, x_2^c, a_0^c, a_1^c, a_2^c, \tau_0, \tau_1, \tau_2, r_0, r_1, r_2).$$

Then the central planner's budget constraint set is given by[14]

$$\hat{b}^c(p,q,\hat{x},e) = \{y^c \in Y^c : p_t x_t^c + q_t a_t^c \leq p_t r_t a_{t-1}^c + p_t \tau_t \hat{x}_t^+(p,q,\tau,r,e)\ \forall t \in T\}$$

where x_t^c is the social planner expenditure at date t and $\hat{x} = (\hat{x}^i)_{i\in I}$ is the market demand strategy.[15] The social planner expenditure given in units of *numéraire* includes social security and public health programs which generate a social welfare u^c. Thus $u^c(x^c)$ represents the social benefit[16] from expenses x^c. Note that public asset interest rates $r_t \in \mathbb{R}_+^k$ and the social planner expenditure x_t^c are endogenously defined for each $t \in T$.

Even when a central planner does not necessarily knows the consumption for each fiscal policy, it can estimate the consumer demand based on historic data of prices and demand. However, the central planner's lack of complete information or the multiplicity of equilibrium preclude it to predict which equilibrium price would be implemented by markets. Therefore, we assume that the game implements a pure Nash equilibrium on prices and a Subgame Perfect Nash Equilibrium[17] on the other strategies. The central planner problem is given by

$$v^c(p,q,\hat{x},e) = \max\left\{\sum_{i\in I} \hat{v}^i(p,q,\tau,r,e) + u^c(x^c) : y^c \in \hat{b}^c(p,q,\hat{x},e)\right\}.$$

Then, the optimal fiscal policy of the social planner given a vector of price p and a market best response function $\hat{y}$ is given by

$$\hat{y}^c(p,q,\hat{x},e) = \operatorname{argmax}\left\{\sum_{i\in I} \hat{v}^i(p,q,\tau,r,e) + u^c(x^c) : y^c \in \hat{b}^c(p,q,\hat{x},e)\right\},$$

and $\hat{y}^c = (\hat{x}^c, \hat{a}^c, \hat{\tau}, \hat{r})$ is the social planner optimal strategy. Recall that $\hat{y} = (\hat{y}^j)_{j\in J}$.

3.3 Equilibrium

Write $\mathscr{E} = (u, \beta, e)$ as the vector of the economy primitives. Thus an equilibrium for $\mathscr{E}$ is given by

1. a market price $(\bar{p}, \bar{q}) \in P \times Q$;

[14] Recall that $\hat{x}_t^+ = \sum_{i\in I} \hat{x}_t^i$ and hence $\hat{b}^c$ does not depend on $\hat{x}^c$.
[15] Note that $a^c < 0$ when central planer is a seller.
[16] This benefit could be viewed as a positive externality.
[17] Actually a type of Stackelberg equilibrium in which the central planner plays the role of the leader.

2. a central planner strategy $\hat{y}^c$ and fiscal policy $\bar{y}^c = (\bar{x}^c, \bar{a}^c, \bar{\tau}, \bar{r})$;
3. a demand strategy $\hat{y}^i = (\hat{x}^i, \hat{a}^i)$ and a demand $\bar{y}^i = \left(\bar{x}^i, \bar{a}^i\right)$ for each i;

such that

1. $\bar{y}^i \in \hat{y}^i(\bar{p}, \bar{q}, \bar{\tau}, \bar{r}, e)$ for each $i \in I$ and $\bar{y}^c \in \hat{y}^c(\bar{p}, \bar{q}, \hat{x}, e)$;
2. $\sum_{i\in I} \bar{a}^i + \bar{a}^c = 0$ and $\sum_{j\in J} \bar{x}^j = \sum_{i\in I} e^i$.

4 Main results

First note that the solution of agents' problem comes from maximization of the Lagrangian evaluated at prices p and fiscal policy y^c. Thus consider the following definition

Definition 4.1. *Define the Lagrangian ℓ^i by*[18]

$$\begin{aligned} \ell^i(p, q, \tau, r, e, y^i) = &\sum_{t\in T} u^i_t(x^i_t) + \lambda^i_t(\tau)(p_t r_t a^i_{t-1} + p_t e^i_t) \\ &- \sum_{t\in T} \lambda^i_t(\tau)(q_t a^i_t + (p_t + p_t \tau_t) x^i_t) \end{aligned} \tag{4}$$

where $\lambda^i_t(\tau) \in \mathbb{R}_+$ for all (p, q, τ, r, e).

Using the F.O.C. fo the agent i, we have the following relationship among the marginal rate of each of the three goods.

Lemma 4.2. *Consider ℓ^i the Lagrangian function. Given $(p, q, y, \hat{y})$, we have*

$$\lambda^i_t(\tau)(1 + \tau_{tk})p_{tk} = \beta^t \partial_1 u^i_{kt}(\hat{x}^i_{kt}(p, q, \tau, r, e)) \textit{ for each } (t, k) \in T \times K \tag{5}$$

where $\hat{x}^i$ is Agent i's demand.

Proof. Fix $(k, t) \in K \times T$. The F.O.C. evaluated at $\dot{y}^i = (\dot{x}^i, \dot{a}^i)$, for each unitary increment

$$\mathring{y}^i(k, t) = (\mathring{x}^i(k, t), \mathring{a}^i(k, t)) \in \mathbb{R}^{\mathrm{k}\times\mathrm{t}} \times \mathbb{R}^{\mathrm{t}}$$

with $\mathring{a}^i(k, t) = 0$ implies that

$$\begin{aligned} \partial_6 \ell^i(p, q, \tau, r, e, y^i)(\mathring{y}^i(k, t)) &= \partial_1 u^i(x^i)(\mathring{x}^i(k, t)) \\ &\quad - \sum_{s\in T} \lambda^i_s(\tau)(p_s + p_s\tau_s)\mathring{x}^i_s(k, t) \\ &= \beta^t \partial_1 u^i_{kt}(x^i_{kt}) - \lambda^i_t(\tau)(p_{tk} + p_{tk}\tau_{tk}) \\ &= 0. \end{aligned} \tag{6}$$

□

[18]Note that we do not consider price dependence of λ^i for the sake of simplicity, since the central planner and markets take it as given.

Remark 4.1. Note that since $\partial_1 u^i_{kt}$ is strictly decreasing, it has an inverse $\breve{u}^i_{kt}$. If $\sum_{i\in I} x^i_t = \sum_{i\in I} e^i_t - x^c := e^+_t - x^c$ then the implicit function theorem implies that the demand of the consumption good of the agent i can be written as a differentiable function $\hat{x}^i_{kt} : \mathbb{R}^4_{++} \to \mathbb{R}_+$ with a typical value $\hat{x}^i_{kt}(p_{tk}, q_t, \tau_{tk}, r_{kt}, e^i_{kt})$.

Additionally, using the Walras' law and the envelop theorem we have the following result.

Lemma 4.3. *Consider $\hat{y}^i = (\hat{x}^i, \hat{a}^i)$ Agent i's optimal choice for some $i \in I$. Fix (p, q, τ, r, e) and let ℓ^i be Agent i's Lagrangian. Then we get*

$$\partial_3 \hat{v}^i(p, q, \tau, r, e)(\mathring{\tau}(t, k)) = -\lambda^i(\tau) p_{tk} \hat{x}^i_{kt}(p, q, \tau, r, e) \tag{7}$$

where $\mathring{\tau}(t, k) \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}}$ is an unitary direction for $(k, t) \in K \times T$.

Proof. Given (p, q, τ, r, e), write $y^i = \hat{y}^i(p, q, \tau, r, e)$, $\bar{\lambda}^i_t = \lambda^i_t(\tau)$ and $\dot{\lambda}^i_{tk} = \partial_1 \lambda^i_t(\tau)(\mathring{\tau}(t, k))$ for each $i \in I$ and $(k, t) \in K \times T$. Using the Walras' law we get by (4)

$$\hat{v}^i(p, q, \tau, r, e) = \ell^i(p, q, \tau, r, e, y^i) \text{ for all } \tau \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}}_+.$$

By the envelop theorem[19] applied to (4) given $(t, k) \in T \times K$ we get from the Walras' Law and (5)

$$\begin{aligned}
\partial_3 \hat{v}^i(p, q, \tau, r, e)(\mathring{\tau}(t, k)) &= \partial_3 \ell^i(p, q, \tau, r, e, y^i)(\mathring{\tau}(t, k)) \\
&= \sum_{t\in T} \dot{\lambda}^i_t (p_t r_t a^i_{t-1} + p_t e^i_t - q_t a^i_t - (p_t + p_t \tau_t) x^i) \\
&- \sum_{s\in T} \bar{\lambda}^i_s p_s \mathring{\tau}_s(t, k) x^i_s \\
&= -\lambda^i_t(\tau) p_{tk} x^i_{kt}.
\end{aligned}$$

for an unitary direction $\mathring{\tau}(t, k)$.

□

Definition 4.4. *Define the market available supply $\hat{e}^{\mathrm{m}}$ given a demand strategy profile $\hat{y}$ and a price (p, q) as*

$$\hat{e}^{\mathrm{m}}(p, q, \hat{y}) = \sum_{i\in I} e^i - \hat{x}^c(p, q, \hat{x}, e).$$

Remark 4.2. Notice that if $(p, q, \hat{y})$ is an equilibrium then

$$\hat{e}^{\mathrm{m}}(p, q, \hat{y}) = \sum_{i\in I} \hat{x}^i(p, q, \hat{\tau}(p, q, \hat{x}, e), \hat{r}(p, q, \hat{x}, e), e).$$

Definition 4.5. *Given a unitary direction $\mathring{\tau}(t, k)$ and $(p, q, \hat{y})$ define the* tax elasticity of aggregate demand *$\hat{x}$ of good $k \in K$ at period $t \in T$ as*

$$\epsilon^+_{tk}(p, q, \hat{y}) = -\frac{\partial_3 \hat{x}^+_{kt}\big(p, q, \hat{\tau}(p, q, \hat{x}, e), \hat{r}(p, q, \hat{x}, e), e\big)(\mathring{\tau}(t, k))}{\hat{x}^+_{kt}\big(p, q, \hat{\tau}(p, q, \hat{x}, e), \hat{r}(p, q, \hat{x}, e), e\big)}.$$

[19] Recall that y^i maximizes the Lagrangian function.

The following result describes an important relation obtained from the first order conditions on the central planner problem. First consider the definition of the central planner's Lagrangian.

Definition 4.6. *Define the central planner's Lagrangian as*

$$\begin{aligned}\ell^c(y^c) = &\sum_{i\in I} \hat{v}^i(p,q,\tau,r,e) + u^c(x^c)\\ &+ \sum_{t\in T} \lambda^c_t\big(p_t r_t a^c_{t-1} + p_t\tau_t \hat{x}^+_t(p,q,\tau,r,e) - q_t a^c_t - p_t x^c_t\big).\end{aligned} \tag{8}$$

Using the F.O.C. of (8) and Lemma 4.3, we have the following result.

Lemma 4.7. *Consider $\mathring{y}^c(k,t) = (\mathring{x}^c, \mathring{\tau}(t,k), \mathring{r}, \mathring{a}^c) \in Y^c$ where $\mathring{\tau}(t,k) \in \mathbb{R}^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}}$ is a unitary direction and $(\mathring{x}^c, \mathring{r}, \mathring{a}^c) = 0$. Let ℓ^c be the central planner Lagrangian. Given $\hat{y}$ and $y^c = \hat{y}(p,q,\hat{x},e)$ then*

$$\tau_{tk}\epsilon^+_{tk}(p,q,\hat{y}) = 1 - \frac{\sum_{i\in I}\lambda^i_t(\tau)p_{tk}\hat{x}^i_{kt}(p,q,\tau,r,e)}{\lambda^c_t p_{tk}\hat{x}^+_{kt}(p,q,\tau,r,e)}.$$

for all $(p,q) \in P\times Q$.

Proof. Using Lemma 4.3 for $k \in K$ we get for $y^c = \hat{y}^c(p,q,\hat{x},e)$

$$\begin{aligned}\partial_1\ell^c(y^c)(\mathring{y}^c(k,t)) = &\sum_{s\in T}\lambda^c_s p_s\tau_s\partial_3 x^+_s(p,q,\tau,r,e)(\mathring{\tau}(t,k))\\ &+\sum_{s\in T}\lambda^c_s p_s\mathring{\tau}_s(t,k)\hat{x}^+_s(p,q,\tau,r,e)\\ &-\sum_{i\in I}\lambda^i_t(\tau)p_{tk}\hat{x}^i_{kt}(p,q,\tau,r,e).\end{aligned}$$

Moreover, by Remark 4.1 we get

$$\partial_3 x^+_{\kappa s}(p,q,\tau,r,e)(\mathring{\tau}(t,k)) = 0 \text{ for all } (\kappa,s)\neq(k,t).$$

Therefore,

$$\begin{aligned}\partial_1\ell^c(y^c)(\mathring{y}^c(k,t)) = &\lambda^c_t p_{tk}\tau_{tk}\partial_3\hat{x}^+_{kt}(p,q,\tau,r,e)(\mathring{\tau}(t,k))\\ &+\lambda^c_t p_{tk}\hat{x}^+_{kt}(p,q,\tau,r,e)\\ &-p_{tk}\sum_{i\in I}\lambda^i_t(\tau)\hat{x}^i_{kt}(p,q,\tau,r,e).\end{aligned}$$

Dividing both sides of this equation by $\lambda^c_t p_{tk}\hat{x}^+_{kt}(p,q,\tau,r,e)$ and using that $\partial_1\ell^c(y^c)(\mathring{y}^c(k,t)) = 0$ we conclude the proof.

□

The following result generalizes Ramsey's result for heterogeneous agents, that is, optimal taxes are higher in goods with lower price elasticity of aggregate demand.

Proposition 4.8. *Suppose that $u^i_{kt}(x^i_{kt}) = \alpha_t x^i_{kt}$ and $\tau_{tk} = 0$ for $k = 0$ and $t \in T$. Given a price (p, q), the optimal taxes $\hat{\tau}_{tk}(p, q, \hat{x}, e)$ satisfy for each $\{k, \kappa\} \subset K/\{0\}$ and $t \in T$*

$$\frac{\hat{\tau}_{tk}(p,q)}{\hat{\tau}_{t\kappa}(p,q)} = \frac{\epsilon^+_{t\kappa}(p,q,\hat{y})}{\epsilon^+_{tk}(p,q,\hat{y})}. \tag{9}$$

Proof. Using (5) we have

$$\lambda^i_t(\tau) = \alpha_t \beta^t \text{ for each } (t,k) \in T \times K.$$

Therefore, by Lemma 4.7 we get

$$\tau_{tk}\epsilon^+_{tk}(p,q,\hat{y}) = 1 - \alpha_t \beta^t / \lambda^c_t \text{ for all } (t,k) \in T \times K$$

and hence

$$\frac{\tau_{tk}\epsilon^+_{tk}(p,q,\hat{y})}{\tau_{t\kappa}\epsilon^+_{t\kappa}(p,q,\hat{y})} = 1$$

which concludes the proof.

□

Definition 4.9. *We say that a market κ is scarcer than k at an equilibrium $(p, q, y, \hat{y})$ when*

1. $\epsilon^+_{t\kappa}(p,q,\hat{y}) > \epsilon^+_{tk}(p,q,\hat{y})$
2. $p_{t\kappa}\hat{e}^{\mathrm{m}}_{t\kappa}(p,q,\hat{y}) < p_{tk}\hat{e}^{\mathrm{m}}_{tk}(p,q,\hat{y})$

Theorem 4.10. *Suppose that $u^i_{kt}(x^i_{kt}) = \alpha_t \log(x^i_{kt})$ for $k = 0$ and $t \in T$. Given an equilibrium $(p, q, y\hat{y})$ suppose also that market κ is scarcer than k. Then the optimal taxes $\hat{\tau}$ satisfy $\hat{\tau}_{t\kappa} > \hat{\tau}_{tk}$ for each $t \in T$.*

Proof. Using (5) we have

$$\lambda^i_t(\tau) p_{tk} \hat{x}^i_{kt}(p,q,\tau,r,e) = \alpha_t \beta^t / (1 + \tau_{tk}) \text{ for each } (t,k) \in T \times K.$$

Therefore, by Lemma 4.7 we get

$$\tau_{tk}\epsilon^+_{tk}(p,q,\hat{y}) - 1 = \frac{\mathrm{i}\alpha\beta^t}{(1+\tau_{tk})\lambda^c_t p_{tk}\hat{e}^{\mathrm{m}}(p,q,\hat{y})} \text{ for all } (t,k) \in T \times K$$

and hence

$$\frac{\tau_{tk}\epsilon^+_{tk}(p,q,\hat{y}) - 1}{\tau_{t\kappa}\epsilon^+_{t\kappa}(p,q,\hat{y}) - 1} = \frac{(1+\tau_{t\kappa})p_{t\kappa}\hat{e}^{\mathrm{m}}_{t\kappa}(p,q,\hat{y})}{(1+\tau_{tk})p_{tk}\hat{e}^{\mathrm{m}}_{tk}(p,q,\hat{y})} \text{ for all } (t,k,\kappa) \in T \times K \times K.$$

By assumption,

$$p_{t\kappa}\hat{e}^{\mathrm{m}}_{t\kappa}(p,q,\hat{y}) < p_{tk}\hat{e}^{\mathrm{m}}_{tk}(p,q,\hat{y})$$

and

$$\epsilon^+_{tk}(p,q,\hat{y}) > \epsilon^+_{t\kappa}(p,q,\hat{y})$$

and hence $\tau_{tk} < \tau_{t\kappa}$. Indeed, if $\tau_{tk} \geq \tau_{t\kappa}$ then

$$1 < \frac{\tau_{tk}\epsilon^{+}_{tk}(p,q,\hat{y}) - 1}{\tau_{t\kappa}\epsilon^{+}_{t\kappa}(p,q,\hat{y}) - 1} = \frac{(1+\tau_{t\kappa})p_{t\kappa}\hat{e}^{\mathrm{m}}_{t\kappa}(p,q,\hat{y})}{(1+\tau_{tk})p_{tk}\hat{e}^{\mathrm{m}}_{tk}(p,q,\hat{y})} < 1$$

which is a contradiction.

□

This result implies that an increment on taxes will affect more lower income agents since a larger proportion of their consumption is spent on the most inelastic good.

4.3 Indeterminacy of the social planner default

After a public health emergency as the COVID-19, central planners must be forced to increase their public health and social security expenses to reduce a massive short-run welfare loss. However, this might increase fiscal fragility making some central planners more willing to implement a strategic default in the long-run.

In this subsection, let us suppose that the central planner is allowed to partial default on its debt at date $t = 2$ ("long-run"), that is, instead of having d_2 as returns at date 2, the return of the public asset is given by δd_2 with $\delta \in [0, 1)$.

In this case, there is no efficiency loss (Mendoza and Yue, 2012) or utility punishment Dubey et al. (2005) to the central planner's default. Therefore, the central planner will have more incentives to default if we compare with a model with an explicit punishment. However, it does not mean that there are no consequences for the central planner's decision of producing a default. The market will take into account the risk that the central planner takes in this case.

The following results shows that if the social planner defaults at $t = 2$, the asset price is adjusted. Moreover, the real allocation implemented without default is also implemented with default.

Definition 4.11. *Let $\delta \in \mathbb{R}^{\mathrm{t\times n\times n}}_{+}$ and $z \in \mathbb{R}^{\mathrm{n\times t}}_{+}$. Define $\delta z \in \mathbb{R}^{\mathrm{n\times t}}_{+}$ as $\delta z = [\delta_t z_t]_{t\in T}$. For $z \in \mathbb{R}^{\mathrm{t\times n}}_{+}$ define $z\delta \in \mathbb{R}^{\mathrm{t\times n}}_{+}$ by $z\delta = (z_t\delta_t)_{t\in T}$. Moreover, define $\delta^{-1} \in \mathbb{R}^{\mathrm{t\times n\times n}}_{+}$ as $\delta^{-1} = (\delta_t^{-1})_{t\in T}$.*

Definition 4.12. *We say that preferences represented by $u^i : X^i \subset \mathbb{R}^{\mathrm{k\times t}}_{+} \to \mathbb{R}$ are semi-radial when for each given diagonal $\delta \in \mathbb{R}^{\mathrm{t\times k\times k}}_{+}$*

$$u^i(\dot{x}^i) \geq u^i(\ddot{x}^i) \Leftrightarrow u^i(\delta\dot{x}^i) \geq u^i(\delta\ddot{x}^i) \text{ for each } (\dot{x}^i, \ddot{x}^i) \in X^i \times X^i.$$

Definition 4.13. *We say that $\delta \in \mathbb{R}^{\mathrm{t\times n\times n}}_{+}$ is an exogenous shock with magnitude $m \in \mathbb{R}^{\mathrm{t}}_{+}$ when $\delta_t = m_t \times I_d$ for each $t \in T$ where I_d is the* $\mathrm{n}\times\mathrm{n}$ *identity matrix.*

Lemma 4.14. *Assume that preferences are semi-radial, and let $\delta \in \mathbb{R}^{\mathrm{t\times k\times k}}_{+}$ be an exogenous shock. Then for each $i \in I$ and each (p, q, τ, r, e)*

$$\begin{aligned} \delta\hat{x}^i(p,q,\tau,r,e) &= \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e) \\ \hat{a}^i(p,q,\tau,r,e) &= \hat{a}^i(p\delta^{-1}, q, \tau, \delta r, \delta e) \end{aligned} \tag{10}$$

Proof. See Lemma 6.1 in the Appendix.

□

Lemma 4.15. *Assume that preferences are semi-radial, then we have for each* $(p, q, \hat{x}, e)$

$$\begin{aligned} \delta \hat{x}^c(p, q, \hat{x}, e) &= \hat{x}^c(p\delta^{-1}, q, \hat{x}, \delta e) \\ \hat{a}^c(p, q, \hat{x}, e) &= \hat{a}^c(p\delta^{-1}, q, \hat{x}, \delta e) \\ \hat{\tau}(p, q, \hat{x}, e) &= \hat{\tau}(p\delta^{-1}, q, \hat{x}, \delta e) \\ \delta \hat{r}(p, q, \hat{x}, e) &= \hat{r}(p\delta^{-1}, q, \hat{x}, \delta e). \end{aligned} \tag{11}$$

Proof. See Lemma 6.2 in the Appendix.

□

Theorem 4.16. *Consider* $(p, q, y, \hat{y})$ *an equilibrium of the economy* $\mathscr{E} = (u, \beta, e)$*. Define*

1. $\bar{p} = p\delta^{-1}$ *and* $\bar{q} = q$
2. $\bar{y}^i = (\bar{x}^i, \bar{a}^i) = (\delta x^i, a^i)$ *for all* $i \in I$
3. $\bar{y}^c = (\bar{x}^c, \bar{a}^c, \bar{\tau}, \bar{r}) = (\delta x^c, a^c, \tau, \delta r)$.

Then $(\bar{p}, \bar{q}, \bar{y}, \hat{y})$ *an equilibrium of the economy* $\mathscr{E} = (u, \beta, \delta e)$

Proof. By (10) we have

$$\bar{x}^i = \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e) \text{ and } \bar{a}^i = \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$$

and by (11)

$$\begin{aligned} \bar{x}^c &= \hat{x}^c(p\delta^{-1}, q, \hat{x}, \delta e) \\ \bar{\tau} &= \hat{\tau}(p\delta^{-1}, q, \hat{x}, \delta e) \\ \bar{a}^c &= \hat{a}^c(p\delta^{-1}, q, \hat{x}, \delta e) \\ \bar{r} &= \hat{r}(p\delta^{-1}, q, \hat{x}, \delta e). \end{aligned} \tag{12}$$

Thus

1. $\bar{y}^i \in \hat{y}^i(\bar{p}, \bar{q}, \bar{\tau}, \bar{r}, \delta e)$ for each $i \in I$ and $\bar{y}^c \in \hat{y}^c(\bar{p}, \bar{q}, \hat{x}, \delta e)$;
2. $\sum_{i\in I} \bar{a}^i + \bar{a}^c = 0$ and $\sum_{i\in I} \bar{x}^i + \bar{x}^c = \sum_i \delta x^i + \delta \bar{x}^c = \sum_i \delta e^i$.

□

The results below make it possible to analyze two environments with and without the shock on aggregate endowments. We can conclude that in an environment with a shock, at the equilibrium the central planner must choose a reduction in sovereign debt relative to the equilibrium without an aggregate shock. Furthermore, a reduction in sovereign debt in the initial period leads to an increase in asset price. This leads the effective interest rate to be endogenously reduced in next period, allowing markets to adjust while maintaining efficiency.

Lemma 4.17. *Assume that preferences are semi-radial. Consider* $\delta \in \mathbb{R}_+^{\mathrm{t\times k\times k}}$ *and* $\gamma \in \mathbb{R}_+^{\mathrm{t\times n\times n}}$ *for* $\mathrm{n} = 1$ *exogenous shocks with the same magnitude.*[20] *Then for each* $i \in I$ *and each* (p, q, τ, r, e)

$$\begin{aligned} \delta\hat{x}^i(p, q, \tau, r, e) &= \hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \\ \gamma\hat{a}^i(p, q, \tau, r, e) &= \hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \end{aligned} \tag{13}$$

Proof. See Lemma 6.3 in the Appendix. □

Lemma 4.18. *Assume that preferences are semi-radial, then we have for each* $(p, q, \hat{x}, e)$

$$\begin{aligned} \delta\hat{x}^c(p, q, \hat{x}, e) &= \hat{x}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \gamma\hat{a}^c(p, q, \hat{x}, e) &= \hat{a}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \hat{\tau}(p, q, \hat{x}, e) &= \hat{\tau}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \hat{r}(p, q, \hat{x}, e) &= \hat{r}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e). \end{aligned} \tag{14}$$

Proof. See Lemma 6.4 in the Appendix. □

Theorem 4.19. *Consider* $(p, q, y, \hat{y})$ *an equilibrium of the economy* $\mathscr{E} = (u, \beta, e)$*. Define*

1. $\bar{p} = p\delta^{-1}$ *and* $\bar{q} = q\gamma^{-1}$
2. $\bar{y}^i = (\bar{x}^i, \bar{a}^i) = (\delta x^i, \gamma a^i)$ *for all* $i \in I$
3. $\bar{y}^c = (\bar{x}^c, \bar{a}^c, \bar{\tau}, \bar{r}) = (\delta x^c, \gamma a^c, \tau, r)$.

Then $(\bar{p}, \bar{q}, \bar{y}, \hat{y})$ *an equilibrium of the economy* $\mathscr{E} = (u, \beta, \delta e)$

Proof. By (10) we get

$$\bar{x}^i = \hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \text{ and } \bar{a}^i = \hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$$

and by (14)

$$\begin{aligned} \bar{x}^c &= \hat{x}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \bar{a}^c &= \hat{a}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \bar{\tau} &= \hat{\tau}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \bar{r} &= \hat{r}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e). \end{aligned} \tag{15}$$

Thus

1. $\bar{y}^i \in \hat{y}^i(\bar{p}, \bar{q}, \bar{\tau}, \bar{r}, \delta e)$ for each $i \in I$ and $\bar{y}^c \in \hat{y}^c(\bar{p}, \bar{q}, \hat{x}, \delta e)$;
2. $\sum_{i\in I} \bar{a}^i + \bar{a}^c = 0$ and $\sum_{i\in I} \bar{x}^i + \bar{x}^c = \sum_i \delta x^i + \delta\bar{x}^c = \sum_i \delta e^i$.

[20] Note that $\delta_t z = z\gamma_t$ for $z \in \mathbb{R}^{\mathrm{k\times n}}$

□

Note that markets anticipate the insolvency of the central planner adjusting the equilibrium price to clear the demand of the asset. Then, the central planner does not obtain any advantage from the default of its debt since the real demand of the consumers and the central planner budget constrained are unchanged due to the "homogeneity of degree zero".

Given the central planner optimal fiscal policy $(\bar{\tau}, \bar{a})$ with full commitment and equilibrium price $(\bar{p}, \bar{q})$, the central planner optimal strategy without full commitment is given by $(\bar{\tau}, \bar{a})$ under the new equilibrium price $(\bar{p}, q_0, \delta\bar{q}_1)$. Normalizing the economy by asset price on the initial period we get that in the new equilibrium, consumption price proportionally increases by an amount of $1/\delta$, and hence the only effect on the economy is an increment of $1/\delta$ on the price of consumption goods at $t = 2$.

Note that in our model a default at $t = 2$ does not have any punishment to the central planner. Therefore, it is indifferent between default its debt partially or pay it completely. In both cases, the central planner does not have incentives to deviate from the optimal taxation system. The only difference is the amount of the public asset available in the market.

However, a default in public bonds has severe consequences in real life economy. Economies with recurrent defaults on their debts face difficulty of finding external currency and private non-speculative investors such as Argentina (Cantamutto and Ozarow, 2016).

Therefore, our result suggests that a central planner must avoid default as much as possible due to the lack of benefits that it will have by doing so. Moreover, it will create two negative signal to the market: the default itself of part of its debt, and a reduction of the public bond price in the short-run. The former is straightforward, the market will naturally cause a bad perception for the investors and hence equilibrium interest rates increase. The latter is a clear signal for investors that the central planner bond is less attractive. At the same time, if the economy uses the public bond to index all prices, there will be a large increment of the price of real goods emulating an 'inflationary' effect due to a high debt and its default in the long-run.

4.4 Income tax and public expenses

Suppose now that the central planner decides to tax only endowments in each market. Assume that agents endowment in each period is their income where e^i_{0t} is the agent i income from the government and e^i_{kt} for $k \in K$ are the agent i income from private sectors. Therefore, the types of taxes considered in this subsection are income taxes.

To obtain the main result of this subsection, we must impose that the marginal utility of a unit of *numéraire* to the type agents ι is lower than the utility that these agents have by consuming it. This is possibly the case of high-paid public employees since their high wages could cause a large fiscal impact and with a low relative social benefit at the margin, or simply public employees with wages above what it is offered in the private sector. Another possibility is that these agents have little impact how the public health crisis can be overcome.

5 Conclusions

We showed that, in a three-period Ramsey's model with heterogeneous agents and a bond offered by the central planner, Ramsey's taxation property holds, that is, when a central planner maximizes a utilitarian welfare function, optimal taxes are higher for markets with less price elasticity. Therefore, increments on the central planner expenditures will impact more the inelastic markets than the elastic ones making the poorest agents of the economy the ones that will pay a larger amount of an extra expenditure.

We also show that a default of part of the central planner bond has no effect on the real economy. Therefore, an increment on public expenditure due to a public health crisis such as the COVID-19 cannot be financed by causing default on the public debt, it will be eventually paid mostly by agents that consume more the inelastic goods, that is, the poorest agents of the economy. Moreover, the central planner will face difficulties to finance its debt due to the type of signals that are given to the market. One of them is the lack of central planner commitment to pay its debt. The other one is the reduction of the public asset price which implies a large increment of the price of real goods in the case of these prices are indexed on the public asset emulating an 'inflationary' effect of the real goods.

Finally, we also show that, with income taxes and public employees, a taxation that is levied on public sector which is not engaged in fighting pandemics can optimally finance these emergency expenditures if the social benefit from having these high-paid employees does not compensate their wages.

6 Appendix

Lemma 6.1. *Assume that preferences are semi-radial, and let $\delta \in \mathbb{R}_+^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}}$ be an exogenous shock. Then for each $i \in I$ and each (p, q, τ, r, e)*

$$\begin{aligned} \delta\hat{x}^i(p, q, \tau, r, e) &= \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e) \\ \hat{a}^i(p, q, \tau, r, e) &= \hat{a}^i(p\delta^{-1}, q, \tau, \delta r, \delta e) \end{aligned} \tag{16}$$

Proof. First we show that

$$\delta\hat{x}^i(p, q, \tau, r, e) \subset \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e).$$

Actually, let $(x^i, a^i) \in \hat{y}^i(p, q, \tau, r, e)$ and $(\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$ arbitrary. Then $(x^i, a^i) \in \hat{b}^i(p, q, \tau, r, e)$ and hence[21]

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\delta_t x_t^i + q_t a_t^i \leq p_t\delta_t^{-1}\delta_t r_t a_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T$$

that is, $(\delta x^i, a^i) \in \hat{b}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$. Moreover,

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\dot{x}_t^i + q_t\dot{a}_t^i \leq p_t\delta_t^{-1}\delta_t r_t\dot{a}_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T.$$

and hence

$$(p_t + p_t\tau_t)\delta_t^{-1}\dot{x}_t^i + q_t\dot{a}_t^i \leq p_t r_t\dot{a}_{t-1}^i + p_t e_t^i \text{ for all } t \in T$$

which implies $(\delta^{-1}\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p, q, \tau, r, e)$. Thus $u^i(x^i) \geq u^i(\delta^{-1}\dot{x}^i)$, that is, $u^i(\delta x^i) \geq u^i(\dot{x}^i)$. Since $(\dot{x}^i, \dot{a}^i)$ was chosen arbitrarily and $x^i \in \hat{x}^i(p, q, \tau, r, e)$, this is the same to state that $\delta\hat{x}^i(p, q, \tau, r, e) \subset \hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$. In addition, $\hat{a}^i(p, q, \tau, r, e) \subset \hat{a}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$

Conversely, let $(x^i, a^i) \in \hat{y}^i(p\delta^{-1}, q, \tau, r\delta, \delta e)$ and $(\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p, q, \tau, r, e)$ arbitrary. Then $(x^i, a^i) \in \hat{b}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$ and hence

$$(p_t + p_t\tau_t)\delta_t^{-1}x_t^i + q_t a_t^i \leq p_t\delta_t^{-1}\delta_t r_t a_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T$$

that is, $(\delta^{-1}x^i, a^i) \in \hat{b}^i(p, q, \tau, r, e)$. Moreover,

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\delta_t\dot{x}_t^i + q_t\dot{a}_t^i \leq p_t\delta_t^{-1}\delta_t r_t\dot{a}_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T$$

that is, $(\delta\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)$ and hence $u^i(x^i) \geq u^i(\delta\dot{x}^i)$, that is, $u^i(\delta^{-1}x^i) \geq u^i(\dot{x}^i)$. Since $(\dot{x}^i, \dot{a}^i)$ was chosen arbitrarily and $x^i = \delta\delta^{-1}x^i$, this is the same to state that $\delta^{-1}x^i \in \hat{x}^i(p, q, r, e)$ and hence $x^i \in \delta\hat{x}^i(p, q, r, e)$.

□

[21] Recall that τ_t is a diagonal matrix and hence it commutes with any matrix in $\mathbb{R}^{\mathrm{k}\times\mathrm{k}}$.

Lemma 6.2. *Assume that preferences are semi-radial, then we have for each* $(p, q, \hat{x}, e)$

$$\begin{aligned}\delta\hat{x}^c(p, q, \hat{x}, e) &= \hat{x}^c(p\delta^{-1}, q, \hat{x}, \delta e)\\ \hat{a}^c(p, q, \hat{x}, e) &= \hat{a}^c(p\delta^{-1}, q, \hat{x}, \delta e)\\ \hat{\tau}(p, q, \hat{x}, e) &= \hat{\tau}(p\delta^{-1}, q, \hat{x}, \delta e)\\ \delta\hat{r}(p, q, \hat{x}, e) &= \hat{r}(p\delta^{-1}, q, \hat{x}, \delta e).\end{aligned} \tag{17}$$

Proof. Analogously to Lemma 6.1, let $y^c \in \hat{y}^c(p, q, \hat{x}, e)$ and $\dot{y}^c \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$ arbitrary. Recall that $y^c = (x^c, a^c, \tau, r)$ and $\dot{y}^c = (\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r})$. Then $y^c \in \hat{b}^c(p, q, \hat{x}, e)$ and hence by (16)

$$\begin{aligned}p_t\delta_t^{-1}\delta x_t^c + q_t a_t^c &\leq p_t\delta_t^{-1}\delta_t r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\delta_t\hat{x}_t^+(p, q, \tau, r, e)\\ &= p_t\delta_t^{-1}\delta_t r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\hat{x}_t^+(p\delta^{-1}, q, \tau, \delta r, \delta e)\end{aligned}$$

for all $t \in T$, that is, $(\delta x^c, a^c, \delta r, \tau) \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$. Since $(\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$ then, for each $t \in T$ by (16) again

$$\begin{aligned}p_t\delta_t^{-1}\dot{x}_t^c + q_t\dot{a}_t^c &\leq p_t\delta_t^{-1}\dot{r}_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\hat{x}_t^+(p\delta^{-1}, q, \dot{\tau}, \dot{r}, \delta e)\\ &= p_t\delta^{-1}\dot{r}_t\dot{a}_{t-1}^c + p_t\dot{\tau}_t\delta_t^{-1}\hat{x}_t^+(p\delta^{-1}, q, \dot{\tau}, \delta\delta^{-1}\dot{r}_t, \delta e)\\ &= p_t\delta^{-1}\dot{r}_t\dot{a}_{t-1}^c + p_t\dot{\tau}_t\hat{x}_t^+(p, q, \dot{\tau}, \delta^{-1}\dot{r}_t, e).\end{aligned}$$

which implies $(\delta_t^{-1}\dot{x}_t^c, \dot{a}^c, \dot{\tau}, \delta^{-1}\dot{r}_t) \in \hat{b}^c(p, q, \hat{x}, e)$. Thus

$$\sum_{i\in I}\hat{u}^i(\hat{x}^i(p, q, \tau, r, e)) + u^c(x^c) \geq \sum_{i\in I}\hat{u}^i(\hat{x}^i(p, q, \dot{\tau}, \delta^{-1}\dot{r}_t, e)) + u^c(\delta_t^{-1}\dot{x}_t^c)$$

and hence

$$\sum_{i\in I}\hat{u}^i(\delta\hat{x}^i(p, q, \tau, r, e)) + u^c(\delta x^c) \geq \sum_{i\in I}\hat{u}^i(\delta\hat{x}^i(p, q, \dot{\tau}, \delta^{-1}\dot{r}_t, e)) + u^c(\delta\delta_t^{-1}\dot{x}_t^c).$$

Therefore,

$$\sum_{i\in I}\hat{u}^i(\hat{x}^i(p\delta^{-1}, q, \tau, \delta r, \delta e)) + u^c(\delta x^c) \geq \sum_{i\in I}\hat{u}^i(\hat{x}^i(p\delta^{-1}, q, \dot{\tau}, \dot{r}, \delta e)) + u^c(\dot{x}^c).$$

Since $\dot{y}^i$ was chosen arbitrarily, we conclude that $(\delta x^c, a^c, \tau, \delta r) \in \hat{y}^c(p\delta^{-1}, q, \hat{x}, \delta e)$, that is,

$$\begin{aligned}\delta\hat{x}^c(p, q, \hat{x}, e) &\subset \hat{x}^c(p\delta^{-1}, q, \hat{x}, \delta e)\\ \hat{a}^c(p, q, \hat{x}, e) &\subset \hat{a}^c(p\delta^{-1}, q, \hat{x}, \delta e)\\ \hat{\tau}(p, q, \hat{x}, e) &\subset \hat{\tau}(p\delta^{-1}, q, \hat{x}, \delta e)\\ \delta\hat{r}(p, q, \hat{x}, e) &\subset \hat{r}(p\delta^{-1}, q, \hat{x}, \delta e).\end{aligned} \tag{18}$$

Conversely, let $y^c \in \hat{y}^c(p\delta^{-1}, q, \hat{x}, \delta e)$ and $\dot{y}^c \in \hat{b}^c(p, q, \hat{x}, e)$ arbitrary. Write $y^c = (x^c, a^c, \tau, r)$, and $\dot{y}^c = (\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r})$. Then $y^c \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$ and hence by (16)

$$\begin{aligned} p_t\delta_t^{-1}x_t^c + q_t a_t^c &\leq p_t\delta_t^{-1}r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\hat{x}_t^+(p\delta_t^{-1}, q, \tau, r, \delta e) \\ &= p_t\delta^{-1}r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\hat{x}_t^+(p\delta^{-1}, q, \tau, \delta\delta^{-1}r, \delta e) \\ &= p_t\delta^{-1}r_t a_{t-1}^c + p_t\tau_t\hat{x}_t^+(p, q, \tau, \delta^{-1}r, e) \end{aligned}$$

for all $t \in T$, that is, $(\delta^{-1}x^c, a^c, \tau, \delta^{-1}r) \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$. Since $(\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p, q, \hat{x}, e)$ then, for each $t \in T$ by (16) again

$$\begin{aligned} p_t\delta_t^{-1}\delta_t\dot{x}_t^c + q_t\dot{a}_t^c &\leq p_t\delta_t^{-1}\delta_t\dot{r}_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\delta_t\hat{x}_t^+(p, q, \dot{\tau}, \dot{r}, e) \\ &= p_t\delta^{-1}\delta_t\dot{r}_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\hat{x}_t^+(p\delta_t^{-1}, q, \dot{\tau}, \delta\dot{r}_t, \delta e). \end{aligned}$$

which implies $(\delta\dot{x}^c, \dot{a}^c, \dot{\tau}, \delta\dot{r}) \in \hat{b}^c(p\delta^{-1}, q, \hat{x}, \delta e)$. Thus

$$\sum_{i\in I}\hat{u}^i(\hat{x}^i(p\delta^{-1}, q, \tau, r, \delta e)) + u^c(x^c) \geq \sum_{i\in I}\hat{u}^i(\hat{x}^i(p\delta^{-1}, q, \dot{\tau}, \delta\dot{r}, \delta e)) + u^c(\delta\dot{x}^c)$$

and hence

$$\sum_{i\in I}\hat{u}^i(\hat{x}^i(p, q, \tau, \delta^{-1}r, e)) + u^c(\delta^{-1}x^c) \geq \sum_{i\in I}\hat{u}^i(\hat{x}^i(p, q, \dot{\tau}, \dot{r}, e)) + u^c(\dot{x}^c).$$

which implies that $(\delta^{-1}x^c, a^c, \delta^{-1}r, \tau) \in \hat{y}^c(p, q, \hat{x}, e)$. This is the same to state that Equations (18) hold in the opposite way.

$\square$

Lemma 6.3. *Assume that preferences are semi-radial and write* $\mathrm{n} = 1$. *Consider* $\delta \in \mathbb{R}_+^{\mathrm{t}\times\mathrm{k}\times\mathrm{k}}$ *and* $\gamma \in \mathbb{R}_+^{\mathrm{t}\times\mathrm{n}\times\mathrm{n}}$ *exogenous shocks with the same magnitude.*[22] *Then for each* $i \in I$ *and each* (p, q, τ, r, e)

$$\begin{aligned} \delta\hat{x}^i(p, q, \tau, r, e) &= \hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \\ \gamma\hat{a}^i(p, q, \tau, r, e) &= \hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \end{aligned} \tag{19}$$

Proof. First we show that

$$\begin{aligned} \delta\hat{x}^i(p, q, \tau, r, e) &\subset \hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \\ \gamma\hat{a}^i(p, q, \tau, r, e) &\subset \hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e). \end{aligned}$$

Actually, let $(x^i, a^i) \in \hat{y}^i(p, q, \tau, r, e)$ and $(\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$ arbitrary. Then $(x^i, a^i) \in \hat{b}^i(p, q, \tau, r, e)$ and hence[23]

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\delta_t x_t^i + q_t\gamma_t^{-1}\gamma_t a_t^i \leq p_t\delta_t^{-1}\delta_t r_t a_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T$$

[22]Note that $\delta_t z_t = z_t\gamma_t$ and hence $\delta_t^{-1}z_t = z_t\gamma_t^{-1}$ for all column vector $z_t \in \mathbb{R}^{\mathrm{k}}$.
[23]Recall that τ_t is a diagonal matrix and hence commutes with any matrix in $\mathbb{R}^{\mathrm{k}\times\mathrm{k}}$.

that is,

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\delta_t x_t^i + q_t\gamma_t^{-1}\gamma_t a_t^i \le p_t\delta_t^{-1} r_t\gamma_t a_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T.$$

This is the same as state that $(\delta x^i, \gamma a^i) \in \hat{b}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$. Moreover,

$$\begin{aligned}(p_t + p_t\tau_t)\delta_t^{-1}\dot{x}_t^i + q_t\gamma_t^{-1}\dot{a}_t^i &\le p_t\delta_t^{-1} r_t\dot{a}_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \\ &= p_t r_t\gamma_t^{-1}\dot{a}_{t-1}^i + p_t e_t^i\end{aligned}$$

for all $t \in T$ which implies $(\delta^{-1}\dot{x}^i, \gamma^{-1}\dot{a}^i) \in \hat{b}^i(p, q, \tau, r, e)$. Thus $u^i(x^i) \ge u^i(\delta^{-1}\dot{x}^i)$, that is, $u^i(\delta x^i) \ge u^i(\dot{x}^i)$. Since $(\dot{x}^i, \dot{a}^i)$ was chosen arbitrarily and $x^i \in \hat{x}^i(p, q, \tau, r, e)$, this is the same to state that

$$\delta\hat{x}^i(p, q, \tau, r, e) \subset \hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$$

and

$$\gamma\hat{a}^i(p, q, \tau, r, e) \subset \hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e).$$

Conversely, let $(x^i, a^i) \in \hat{y}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$ and $(\dot{x}^i, \dot{a}^i) \in \hat{b}^i(p, q, \tau, r, e)$ arbitrary. Then $(x^i, a^i) \in \hat{b}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$ and hence

$$(p_t + p_t\tau_t)\delta_t^{-1} x_t^i + q_t\gamma_t^{-1} a_t^i \le p_t r_t\gamma_t^{-1} a_{t-1}^i + p_t e_t^i \text{ for all } t \in T$$

that is, $(\delta^{-1}x^i, \gamma^{-1}a^i) \in \hat{b}^i(p, q, \tau, r, e)$. Moreover,

$$(p_t\delta_t^{-1} + p_t\delta_t^{-1}\tau_t)\delta_t\dot{x}_t^i + q_t\gamma_t^{-1}\gamma_t\dot{a}_t^i \le p_t\delta_t^{-1} r_t\gamma_t\dot{a}_{t-1}^i + p_t\delta_t^{-1}\delta_t e_t^i \text{ for all } t \in T$$

that is, $(\delta\dot{x}^i, \gamma\dot{a}^i) \in \hat{b}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)$ and hence $u^i(x^i) \ge u^i(\delta\dot{x}^i)$, that is, $u^i(\delta^{-1}x^i) \ge u^i(\dot{x}^i)$. Since $(\dot{x}^i, \dot{a}^i)$ was chosen arbitrarily, this is the same to state that $(\delta^{-1}x^i, \gamma^{-1}a^i) \in \hat{y}^i(p, q, r, e)$. Therefore $\hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \subset \delta\hat{x}^i(p, q, \tau, r, e)$ and $\hat{a}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \subset \delta\hat{a}^i(p, q, \tau, r, e)$ since $(x^i, a^i) = (\delta\delta^{-1}x^i, \gamma\gamma^{-1}a^i)$.

□

Lemma 6.4. *Assume that preferences are semi-radial, then we have for each* $(p, q, \hat{x}, e)$

$$\begin{aligned}\delta\hat{x}^c(p, q, \hat{x}, e) &= \hat{x}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \gamma\hat{a}^c(p, q, \hat{x}, e) &= \hat{a}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e) \\ \hat{\tau}(p, q, \hat{x}, e) &= \hat{\tau}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e) \\ \hat{r}(p, q, \hat{x}, e) &= \hat{r}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e).\end{aligned} \tag{20}$$

Proof. Analogously to Lemma 6.1, let $y^c \in \hat{y}^c(p, q, \hat{x}, e)$ and $\dot{y}^c \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$ arbitrary. Then $y^c \in \hat{b}^c(p, q, \hat{x}, e)$ and hence by (19)

$$\begin{aligned}p_t\delta_t^{-1}\delta x_t^c + q_t\gamma_t^{-1}\gamma_t a_t^c &\le p_t\delta_t^{-1}\delta_t r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\delta_t\hat{x}_t^+(p, q, \tau, r, e) \\ &= p_t\delta_t^{-1} r_t\gamma_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\hat{x}_t^+(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)\end{aligned}$$

for all $t \in T$, that is, $(\delta x^c, \gamma a^c, \tau, r) \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$. Since $(\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$ then, for each $t \in T$ by (19) again

$$\begin{aligned} p_t\delta_t^{-1}\dot{x}_t^c + q_t\gamma_t^{-1}\dot{a}_t^c &\leq p_t\delta_t^{-1}\dot{r}_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\hat{x}_t^+(p\delta^{-1}, q\gamma^{-1}, \dot{\tau}, \dot{r}, \delta e) \\ &= p_t\dot{r}_t\gamma_t^{-1}\dot{a}_{t-1}^c + p_t\dot{\tau}_t\delta_t^{-1}\delta_t\hat{x}_t^+(p, q, \dot{\tau}, \dot{r}, \delta e) \\ &= p_t\dot{r}_t\gamma^{-1}\dot{a}_{t-1}^c + p_t\dot{\tau}_t\hat{x}_t^+(p, q, \dot{\tau}, \dot{r}, e). \end{aligned}$$

which implies $(\delta^{-1}\dot{x}^c, \gamma^{-1}\dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p, q, \hat{x}, e)$. Thus

$$\sum_{i \in I} \hat{u}^i(\hat{x}^i(p, q, \tau, r, e)) + u^c(x^c) \geq \sum_{i \in I} \hat{u}^i(\hat{x}^i(p, q, \dot{\tau}, \dot{r}, e)) + u^c(\delta^{-1}\dot{x}^c)$$

and hence

$$\sum_{i \in I} \hat{u}^i(\delta\hat{x}^i(p, q, \tau, r, e)) + u^c(\delta x^c) \geq \sum_{i \in I} \hat{u}^i(\delta\hat{x}^i(p, q, \dot{\tau}, \dot{r}, e)) + u^c(\delta\delta^{-1}\dot{x}^c).$$

Therefore,

$$\sum_{i \in I} \hat{u}^i(\hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)) + u^c(\delta x^c) \geq \sum_{i \in I} \hat{u}^i(\hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \dot{\tau}, \dot{r}, \delta e)) + u^c(\dot{x}^c).$$

which implies $(\delta x^c, \gamma a^c, \tau, r) \in \hat{y}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$. We conclude that

$$\begin{aligned} \delta\hat{x}^c(p, q, \hat{x}, e) &\subset \hat{x}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e) \\ \gamma\hat{a}^c(p, q, \hat{x}, e) &\subset \hat{a}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e) \\ \hat{\tau}(p, q, \hat{x}, e) &\subset \hat{\tau}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e) \\ \hat{r}(p, q, \hat{x}, e) &\subset \hat{r}(p\delta^{-1}, q\gamma^{-1}, \hat{x}, e). \end{aligned} \tag{21}$$

Conversely, let $y^c \in \hat{y}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$ and $\dot{y}^c \in \hat{b}^c(p, q, \hat{x}, e)$ arbitrary. Write $y^c = (x^c, a^c, \tau, r)$, and $\dot{y}^c = (\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r})$. Then $y^c \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$ and hence by (19)

$$\begin{aligned} p_t\delta_t^{-1}x_t^c + q_t\gamma_t^{-1}a_t^c &\leq p_t\delta_t^{-1}r_t a_{t-1}^c + p_t\delta_t^{-1}\tau_t\hat{x}_t^+(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e) \\ &= p_t r_t\gamma_t^{-1}a_{t-1}^c + p_t\delta_t^{-1}\tau_t\delta_t\hat{x}_t^+(p, q, \tau, r, e) \\ &= p_t r_t\gamma_t^{-1}a_{t-1}^c + p_t\tau_t\hat{x}_t^+(p, q, \tau, r, e) \end{aligned}$$

for all $t \in T$, that is, $(\delta^{-1}x^c, \gamma^{-1}a^c, r, \tau) \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$. Since $(\dot{x}^c, \dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p, q, \hat{x}, e)$ then, for each $t \in T$ by (19) again

$$\begin{aligned} p_t\delta_t^{-1}\delta_t\dot{x}_t^c + q_t\gamma_t^{-1}\gamma_t\dot{a}_t^c &\leq p_t\delta_t^{-1}\delta_t\dot{r}_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\delta_t\hat{x}_t^+(p, q, \dot{\tau}, \dot{r}, e) \\ &= p_t\delta^{-1}\dot{r}_t\gamma_t\dot{a}_{t-1}^c + p_t\delta_t^{-1}\dot{\tau}_t\hat{x}_t^+(p\delta_t^{-1}, q\gamma_t^{-1}, \dot{\tau}, \dot{r}, \delta e) \end{aligned}$$

which implies $(\delta\dot{x}^c, \gamma\dot{a}^c, \dot{\tau}, \dot{r}) \in \hat{b}^c(p\delta^{-1}, q\gamma^{-1}, \hat{x}, \delta e)$. Thus

$$\sum_{i\in I} \hat{u}^i(\hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \tau, r, \delta e)) + u^c(x^c) \geq \sum_{i\in I} \hat{u}^i(\hat{x}^i(p\delta^{-1}, q\gamma^{-1}, \dot{\tau}, \dot{r}, \delta e)) + u^c(\delta\dot{x}^c)$$

and hence

$$\sum_{i\in I} \hat{u}^i(\hat{x}^i(p, q, \tau, r, e)) + u^c(\delta^{-1}x^c) \geq \sum_{i\in I} \hat{u}^i(\hat{x}^i(p, q, \dot{\tau}, \dot{r}, e)) + u^c(\dot{x}^c).$$

which implies that $(\delta^{-1}x^c, \gamma^{-1}a^c, r, \tau) \in \hat{y}^c(p, q, \hat{x}, e)$. This implies that (21) hold in the opposite way.

$\square$

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